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Asymptotics of Racah coefficients and polynomials*

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Abstract. Ponzano and Regge conjectured certain asymptotic formulae for Racah coefficients with large angular momenta and gave convincing numerical evidence of their validity. We provide rigorous proofs of their asymptotic formulae. We use generating functions to derive an integral representation of Racah polynomials and then apply complex analytic methods to determine the asymptotic behaviour of the Racah coefficients.

1. Introduction

The Racah–Wigner coefficients appeared in a paper by Racah [9] as a tool for the computation of matrix elements in the theory of complex spectra. They have been defined as the transformation coefficients between two different coupling schemes of any three angular momenta and have proved to play an important role in the theories of atomic and nuclear spectroscopy [2]. The asymptotic properties of the angular momentum functions is of interest in any general discussion of angular momentum theory [20, 2, 16].

In 1968, Ponzano and Regge [8] gave heuristic derivations of several asymptotic formulae for the Racah coefficients (or 6-j symbols). They expressed the desire to find a rigorous proof of their conjecture. In 1975, Schulten and Gordon [11] established a three-term recurrence relation satisfied for the 6-j symbol. They noted in [12] that in the semiclassical limit the recurrence relation tends to a differential equation. They then applied the WKB method to the limiting differential equation to determine the asymptotic behaviour of its solution, and concluded that the solution of the original three-term recurrence relation must have the same asymptotic behaviour. This approach, contrary to the claim in [12], is not rigorous. Schulten and Gordon's approach, however, raises the interesting problem of finding a discrete analogue of the Liouville–Green approximation [7] and the WKB method. This question was actually raised in the earlier work of Ponzano and Regge [8]. The currently known discrete versions of the WKB method [5, 6, 14] do not seem to be applicable to the Ponzano–Regge asymptotics.

This work grew out of an attempt to justify the Ponzano–Regge asymptotics. To the best of our knowledge, a direct rigorous proof of the aforementioned asymptotic results has not been found to date. The closest proof to be rigorous is in sections 6–8 of topic 9 in chapter 5 of Biedenharn and Louck's influential book [3]. The proof consists of deriving three fundamental identities that characterize the Racah coefficients, stated as (5.8.3)–(5.8.5), and then shows that the main terms in the Ponzano–Regge conjectured asymptotics satisfy these identities. This is a very interesting proof but in the semiclassical limit under considerations the number of terms

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in the above-mentioned identities (5.8.3)–(5.8.5) depends on the large parameters and more uniform estimates are needed to make the proof completely rigorous. The Biedenharn–Louck proof is very geometric and its authors say that Ponzano and Regge were careful not to claim a rigorous proof for their asymptotic results. Then they go on to say 'In our opinion, their (Ponzano and Regge's) work constitutes the essential elements of a valid proof, and only details (such as the proof of Racah's identity given in section 8) needed explanation'. We feel that Biedenharn–Louck's proof retains many of the geometric approach in [8], but it is extremely modest of Biedenharn and Louck to say that they merely filled in the details. Their proof takes 13 pages.

Our approach uses generating functions and the classical theory of hypergeometric functions. A generalized hypergeometric function ${}_pF_q$ is defined as

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = {}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_p)_k z^k}{(b_1)_k\ldots(b_q)_k k!} \tag{1.1} $$

where the shifted factorial $(a)_k$ is defined by

$$ (a)_k := a(a+1)\cdots(a+k-1), k \in \mathbb{N} \qquad (a)_0 = 1. \tag{1.2} $$

Note that (1.2) is equivalent to

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \tag{1.3} $$

when $k \geqslant 0$, but (1.3) holds for all integers provided that the right-hand side is well defined.

The 3-j and 6-j symbols of quantum mechanics turned out to be multiples of generalized hypergeometric functions. The 3-j symbols $\begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix}$, introduced by Wigner, are defined [4, 2, 3] by

$$ \begin{aligned} \begin{pmatrix} a & b & e \\ d & c & f \end{pmatrix} &= (-1)^{a-b-f}\delta_{d+c,-f}\Delta(a,b,e) \\ &\quad\times[(e-f)!(e+f)!(a+d)!(a-d)!(b+c)!(b-c)!]^{1/2} \\ &\quad\times\sum_s(-1)^s[s!(e-b+d+s)!(e-a-c+s)!(a+b-e-s)! \\ &\quad\times(a-d-s)!(b+c-s)!]^{-1} \end{aligned} \tag{1.4} $$

where $\Delta(a,b,e)$ is defined with (1.6).

The Wigner 6-j symbols $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$ and the Racah coefficients $W(a,b,c,d;e,f)$ are defined (see [2, 3]) by

$$ \begin{aligned} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} &= (-1)^{a+b+c+d}W(a,b,c,d;e,f) = \Delta(a,b,e)\Delta(c,d,e)\Delta(a,c,f)\Delta(b,d,f) \\ &\quad\times\sum_s\frac{(-1)^s(s+1)!}{(s-a-b-e)!(s-c-d-e)!(s-a-c-f)!(s-b-d-f)!} \\ &\quad\times\frac{1}{(a+b+c+d-s)!(a+d+e+f-s)!(b+c+e+f-s)!} \end{aligned} \tag{1.5} $$

where $\Delta(a,b,e)$ is defined (see [2]) by

$$ \Delta(a,b,e) = \left[\frac{(a+b-e)!(a-b+e)!(-a+b+e)!}{(a+b+e+1)!}\right]^{1/2}. \tag{1.6} $$

The Racah polynomials $\{r_n(\lambda(x); \beta, \gamma, \delta)\}$ are defined by

$$r_n(\lambda(x); \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -n, n+\beta-N, -x, x+\gamma+\delta+1 \\ \beta+\delta+1, \gamma+1, -N \end{matrix} \middle| 1 \right) \tag{1.7}$$

$n = 0, 1, \ldots, N$, where

$$\lambda(x) := [x + (\gamma+\delta+1)/2]^2. \tag{1.8}$$

For connection with orthogonal polynomials the reader may consult [1].

We shall write $F(R) \approx G(R)$ if $F(R)/G(R) = 1 + o(1)$ as $R \to \infty$, and $F(R) \approx G(R)$, $\{\phi(R)\}$ if $F(R)/G(R) = 1 + \phi(R)$, where $\phi(R) \to 0$ as $R \to \infty$.

Ponzano and Regge [8] gave a heuristic argument to determine the main term in the asymptotic development of certain 6-j symbols. In sections 2 and 3 we give a rigorous proof of the the following asymptotics formulae predicted by Ponzano and Regge.

Theorem 1. *The Wigner 6-j symbols (1.5) satisfy*

$$\begin{Bmatrix} a & b & e \\ d+R & c+R & f+R \end{Bmatrix} \approx (-1)^{a+b+e}(2R)^{-1/2} \begin{pmatrix} a & b & e \\ c-f & f-d & d-c \end{pmatrix} \qquad \{R^{-1}\} \tag{1.9}$$

and

$$\begin{pmatrix} a & b+R & e+R \\ d & c+R & f+R \end{pmatrix}$$
$$\approx \left\{ \frac{(a+b-e)!(a+c-f)!(d+c-e)!(d+b-f)!}{(a-b+e)!(a-c+f)!(d-c+e)!(d-b+f)!} \right\}^{1/2-\mathrm{sign}(e+f-b-c)}$$
$$\times (-1)^{a+d+\min(b+c,e+f)} \frac{(2R)^{-|b+c-e-f|-1}}{(|b+c-e-f|)!} \qquad \{R^{-1}\} \tag{1.10}$$

as $R \to \infty$.

Theorem 2. *Let a, b, c, and f be large positive integers ($\to \infty$) and $f = o((\min\{a, b, e\})^{1/2})$. Then the 6-$j$ symbols satisfy*

$$\begin{Bmatrix} a & b & e \\ b & a & f \end{Bmatrix} \approx \frac{(-1)^{a+b+e+f}}{\sqrt{(2a+1)(2b+1)}} P_f(\cos\theta) \qquad \{f^2/m\} \tag{1.11}$$

where P_f is the Legendre polynomial of degree f and θ is given by

$$\cos\theta = \frac{(a(a+1) + b(b+1) - e(e+1))}{2\sqrt{a(a+1)b(b+1)}}. \tag{1.12}$$

Furthermore,

$$\begin{Bmatrix} a & b & e \\ b & a & f \end{Bmatrix} \approx \frac{(-1)^{a+b+e+f}}{\sqrt{2\pi(a+1/2)(b+1/2)(f+1/2)\sin\theta}}$$
$$\times \cos\left(\left(f + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right) \qquad \left\{\frac{f^2}{m} + \frac{1}{f^{3/2}}\right\}. \tag{1.13}$$

Besides verifying (1.9) and (1.10) in section 2, and (1.11) and (1.13) in section 3, we also estimate the first error terms. Our methods allow explicit computation of all error terms, in particular we find the first error term in (1.10) to show that it is of $O(R^{-1})$ and not $O(R^{-2})$ as claimed by Ponzano and Regge. Our proof of theorem 2 shows that the condition that $f = O(\min\{a, b, e\})$ of Ponzano and Regge is not sufficient, we rather need $f = o((\min\{a, b, e\})^{1/2})$.

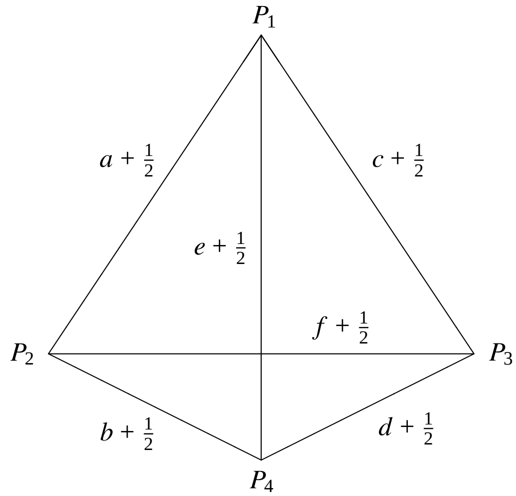

Figure 1. Coupling tetrahedron.

In section 4 we study the asymptotic behaviour, (see [7]), of the class of orthogonal polynomials of a discrete variable defined with (1.7) and (1.8), the Racah polynomials.

The 6-j symbol (1.5) can be written as a multiple of a ${}_4F_3$ expression. Let $\{\alpha_1, \alpha_2\alpha_3, \alpha_4\}$ be any permutation of $\{a+b+e, c+d+e, a+c+f, b+d+f\}$, the perimeters of the four sides of the tetrahedron on figure 1, and let $\alpha_1 = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Let $\{\beta_1, \beta_2, \beta_3\} = \{a+b+c+d, a+d+e+f, b+e+c+f\}$, the numbers of the form $S-x-y$, where (x, y) are the pairs of opposite edges in that tetrahedron, and let $\beta_1 = \min\{\beta_1, \beta_2, \beta_3\}$. To write the sum S in (1.5) in terms of a ${}_4F_3$ we set $s = \alpha_1 + l$, $l \in \{0, 1, \dots, n\}$, where $n = \beta_1 - \alpha_1$. Using the identity $(A-l)! = (-1)^l A!/((-A)_l)$ we get

$$\begin{aligned} S &= \sum_{l=0}^{n} \frac{(-1)^{\alpha_1+l}(\alpha_1+l+1)!}{l!(\alpha_1-\alpha_2+l)!(\alpha_1-\alpha_3+l)!(\alpha_1-\alpha_4+l)!} \\ &\qquad \times \frac{1}{(\beta_1-\alpha_1-l)!(\beta_2-\alpha_1-l)!(\beta_3-\alpha_1-l)!} \\ &= \frac{(-1)^{\alpha_1}(\alpha_1+1)!}{(\alpha_1-\alpha_2)!(\alpha_1-\alpha_3)!(\alpha_1-\alpha_4)!(\beta_1-\alpha_1)!(\beta_2-\alpha_1)!(\beta_3-a_1)!} \\ &\qquad \times {}_4F_3\left(\begin{matrix} \alpha_1+2, \alpha_1-\beta_1, \alpha_1-\beta_2, \alpha_1-\beta_3 \\ \alpha_1-\alpha_2+1, \alpha_1-\alpha_3+1, \alpha_1-\alpha_4+1 \end{matrix} \middle| 1 \right). \end{aligned} \tag{1.14}$$

The 3-j symbols are related to Hahn polynomials. Some interesting asymptotics of the Hahn polynomials are in [19].

We shall use the Chu–Vandermonde sum (the terminating version of Gauss's theorem), Rainville [10],

$${}_2F_1(-n, a; b; 1) = \frac{(b-a)_n}{(b)_n} \tag{1.15}$$

and the Pfaff–Kummer transformation, Rainville [10], Slater [13],

$${}_2F_1(a, b; c; z) = (1-z)^{-a}{}_2F_1(a, c-b; c; z/(z-1)). \tag{1.16}$$

Formula (1.16) is valid for $|z| < 1$ and $\mathrm{Re}(z) < \frac{1}{2}$, or if it terminates.

We shall also use the Whipple transformation [17, 13]

$$
{}_4F_3\left(\begin{matrix} -n, A, B, C \\ D, E, F \end{matrix}\,\middle|\, 1\right)
= \frac{(E-A)_n(F-A)_n}{(E)_n(F)_n}\,{}_4F_3\left(\begin{matrix} -n, A, D-B, D-C \\ D, 1-n+A-E, 1-n+A-F \end{matrix}\,\middle|\, 1\right) \qquad (1.17)
$$

which holds if n is a non-negative integer and $D+E+F=-n+A+B+C$, as well as the asymptotic formula, [7, equation (4.5.02)],

$$
\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1+\frac{(a-b)(a+b-1)}{2z}+\mathrm{O}(z^{-2})\right] \qquad (1.18)
$$

as $z\to\infty$ with $|\arg(z)|\leqslant\pi-\delta<\pi$.

The Whipple transformation was used by Wilson [18] to derive a complete asymptotic expansion for the ${}_4F_3$ in (1.17), as $n\to\infty$, when $-n+A, B, C, D, E$ do not depend on n. The asymptotics treated here correspond to $D=-N$ with $n<N$, so when $n\to\infty$, $N\to\infty$ as well.

2. Proof of theorem 1

The coupling of angular momenta can be represented by the following diagram.

We associate the 6-j symbol $\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$ with figure 1. We are interested in finding approximation formulae for the 6-j symbols when one or more edges of the tetrahedron tend to infinity. We can picture the limiting process as one or more edges tend to infinity by associating it with the partitions of 4, namely, 1 + 3, 2 + 2, 1 + 1 + 2, and 1 + 1 + 1 + 1. If one vertex tends to infinity then, in the tetrahedron, there are three fixed edges and three edges that approach infinity. The four vertices are now divided into two groups in the 1 + 3 case, in this case we may picture both groups tending to infinity but the second group, which contains three vertices, goes to infinity in a cluster. This is essentially the confluence relation (1.9).

2.1. Proof of asymptotic formula (1.10)

This is the 2 + 2 case when all vertices tend to infinity but two nonadjacent edges remain bounded. Our proof of (1.10) shows that in (1.10) $\phi(R)=\mathrm{O}(R^{-1})$ and not $\mathrm{O}(R^{-2})$ as was claimed by Ponzano and Regge [8].

Proposition 2.1. *Assume that $n=b+d-f$ is the integer which determines the number of terms in the ${}_4F_3$ terminating series in (1.14). Then*

$$
\begin{Bmatrix} a & b+R & e+R \\ d & c+R & f+R \end{Bmatrix} = \left\{\frac{(a-b+e)!(a-c+f)!(d-c+e)!(d-b+f)!}{(a+b-e)!(a+c-f)!(d+c-e)!(d+b-f)!}\right\}^{1/2}
\times(-1)^{a+b+c+d}\frac{(2R)^{-(e+f-b-c)-1}}{(e+f-b-c)!}
\times\left(1+\frac{(b+c+1/2)^2-(e+f+3/2)^2}{4R}+\mathrm{O}\left(\frac{1}{R^2}\right)\right). \qquad (2.1)
$$

Proof. As we have shown in section 1, the number of terms in the sum S in (1.5) is $n+1$, where (see (1.14)), $n=\beta_1-\alpha_1$. Since a triangle and a pair of opposite edges have exactly one edge in common, we must have that $\alpha_1 = a + c + f$ and $\beta_1 = a + b + c + d$. By (1.14) for the sum S in (1.5) we have

$$S = p(a, c, f; b, d, e)(-1)^{a+c+f} \times {}_4F_3\left(\begin{matrix} -b-d+f, , a+c+f+2, -d-e+c, -b-e+a \\ c+f-b-e+1, a+f-d-e+1, a+c-b-d+1 \end{matrix} \middle| 1\right)$$

where $p(a, c, f; b, d, e)$ denotes the quantity

$$\frac{(a+c+f+1)!}{(c+f-b-e)!(a+f-d-e)!(a+c-b-d)!} \times [(b+d-f)!(d+e-c)!(b+e-a)!]^{-1}. \tag{2.2}$$

Adding $R \in \mathbb{N}$ to $b, e, c,$ and f does not change the numbers α_1, β_1, and the number of terms in the terminating ${}_4F_3$ in (1.14), hence we obtain

$$\begin{aligned}
\begin{Bmatrix} a & b+R & e+R \\ d & c+R & f+R \end{Bmatrix}
&= \Delta(a, b+R, e+R)\Delta(c+R, d, e+R)\Delta(a, c+R, f+R)\Delta(b+R, d, f+R) \\
&\quad \times {}_4F_3\left(\begin{matrix} -n, a+c+f+2+2R, -d-e+c, -b-e+a-2R \\ c+f-b-e+1, a+f-d-e+1, a+c-b-d+1 \end{matrix} \middle| 1\right) \\
&\quad \times (-1)^{a+c+f+2R} p(a, c+R, f+R; b+R, d, e+R) \\
&= (2R)^{-2a-2d-2}[(a+b-e)!(a-b+e)!(d+e-c)!(d-e+c)! \\
&\quad \times (a+c-f)!(a-c+f)!(d+f-b)!(d-f+b)!]^{1/2} \\
&\quad \times \left(1 - \frac{(a+d+1)(b+c+e+f+2)}{2R} + \mathrm{O}\left(\frac{1}{R^2}\right)\right) \\
&\quad \times \frac{(-c-d-e-1-2R)_n(-b-d-f-1-2R)_n}{(a+f-d-e+1)_n(a+c-b-d+1)_n} {}_4F_3 \\
&\quad \times \left(\begin{matrix} -n, a+c+f+2R+2, d+f-b+1, c+f-a+1+2R \\ c+f-b-e+1, c+d+e+2-n+2R, b+d+f+2-n+2R \end{matrix} \middle| 1\right) \\
&\quad \times (-1)^{a+c+f}(a+c+f+2R+1)![(c+f-b-e)!(a+f-d-e)! \\
&\quad \times (a+c-b-d)!(b+d-f)!(d+e-c)!(b+e-a+2R)!]^{-1}
\end{aligned} \tag{2.3}$$

where we used that for fixed integers x, y, z, and $R \to \infty$,

$$\Delta(x, y+R, z+R) = [(x+y-z)!(x-y+z)!]^{1/2}(2R)^{-x-1/2} \times \left(1 - \frac{(2x+1)(y+z+1)}{4R} + \mathrm{O}\left(\frac{1}{R^2}\right)\right) \tag{2.4}$$

which follows from

$$\frac{(p+M)!}{(q+M)!} = \frac{\Gamma(p+M+1)}{\Gamma(q+M+1)} = M^{p-q}\left(1 + \frac{(p-q)(p+q+1)}{2M} + \mathrm{O}\left(\frac{1}{M^2}\right)\right) \tag{2.5}$$

for fixed integers p, q, and $M \to \infty$ (see (1.18)). We also used the Whipple transformation (1.17) with $A = a+c+f+2+2R$, $B = -d-e+c$, $C = -b-e+a-2R$, $D = c+f-b-e+1$, $E = a+f-d-e+1$, and $F = a+c-b-d+1$.

Next we find the the asymptotics of the last ${}_4F_3$ expression as $R \to \infty$. For fixed numbers $k \in \mathbb{N}$, p and a large M from (1.18) we have

$$(p+M)_k = \frac{\Gamma(p+M+k)}{\Gamma(p+M)} = M^k\left(1 + \frac{k(2p+k-1)}{2M} + \mathrm{O}\left(\frac{1}{M^2}\right)\right). \tag{2.6}$$

Using (2.6) we get

$$\frac{(a+c+f+2+2R)_k(c+f-a+1+2R)_k}{(c+d+e+2-n+2R)_k(b+d+f+2-n+2R)_k} = 1+\frac{k(b+c-e-f-1)}{2R}+\mathrm{O}(R^{-2}). \tag{2.7}$$

Substituting these estimates in the last ${}_4F_3$ and using (4.4), for its asymptotics we get

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}-n,\ d+f-b+1\\ c+f-b-e+1\end{matrix}\,\middle|\,1\right)+\frac{1}{2R}(b+c-e-f-1)\frac{(-n)(d+f-b+1)}{(c+f-b-e+1)}\\
&\qquad\times{}_2F_1\left(\begin{matrix}-n+1,\ d+f-b+2\\ c+f-b-e+2\end{matrix}\,\middle|\,1\right)+\mathrm{O}\left(R^{-2}\right)=\frac{(c-d-e)_n}{(c+f-b-e+1)_n}\\
&\qquad+\frac{1}{2R}\frac{(c-d-e+n-1)n(d+f-b+1)}{(c+f-b-e+1)}\frac{(c-d-e)_{n-1}}{(c+f-b-e+2)_{n-1}}+\mathrm{O}(R^{-2})\\
&\qquad=\frac{(c-d-e)_n}{(c+f-b-e+1)_n}\left(1-\frac{1}{2R}n(d+f-b+1)+\mathrm{O}(R^{-2})\right)\\
&\qquad=\frac{(-1)^n(d+e-c)!}{(e+f-b-c)!}\frac{(c+f-b-e)!}{(c+d-e)!}\left(1-\frac{1}{2R}n(d+f-b+1)+\mathrm{O}(R^{-2})\right)
\end{aligned} \tag{2.8}$$

since $c-d-e<0$, and $d+e-c\geqslant n=b+d-f$ by assumption, where we also used (1.15) and $(\alpha+1)_n=(\alpha+n)!/\alpha!$ for $\alpha\geqslant 0$.

Similarly by (2.6) we get

$$\begin{aligned}
&(-c-d-e-1-2R)_n(-b-d-f-1-2R)_n\\
&\qquad=(c+d+e+2-n+2R)_n(b+d+f+2-n+2R)_n\\
&\qquad=(2R)^{2n}\left(1+\frac{n(c+d+e+2f+3)}{2R}+\mathrm{O}(R^{-2})\right)
\end{aligned} \tag{2.9}$$

and by (2.5) we get

$$\begin{aligned}
\frac{(a+c+f+1+2R)!}{(b+e-a+2R)!}&=(2R)^{2a+c-b+f-e+1}\\
&\quad\times\left(1+\frac{(2a+c+f-b-e+1)(b+c+e+f+2)}{4R}+\mathrm{O}(R^{-2})\right).
\end{aligned} \tag{2.10}$$

We also have

$$(a+f-d-e+1)_n(a+c-b-d+1)_n=\frac{(a+b-e)!(a+c-f)!}{(a+f-d-e)!(a+c-b-d)!}. \tag{2.11}$$

Substituting (2.11), (2.8), (2.9), and (2.10) in (2.3) we obtain (2.1). □

The asymptotic formula (1.10) follows from proposition 2.1 and the relation $\begin{Bmatrix}a & b & e\\ d & c & f\end{Bmatrix}=\begin{Bmatrix}a & e & b\\ d & f & c\end{Bmatrix}$ implied by (1.5).

Proof of asymptotic formula (1.9). This is the 1 + 3 case when all edges tend to infinity with the same rate and three edges approach infinity in a cluster. We actually establish a more general asymptotic formula of which (1.9) is a special case.

From (1.5) we have

$$\begin{aligned}
\begin{Bmatrix}a & b & e\\ d+R & c+R & f+R\end{Bmatrix}&=\Delta(a,b,e)\Delta(c+R,d+R,e)\\
&\quad\times\Delta(a,c+R,f+R)\Delta(b,d+R,f+R)S(R)
\end{aligned} \tag{2.12}$$

where $S(R)$ is the sum from (1.5) with c, d, f replaced by $c+R, d+R, f+R$, respectively. The index of the sum $s = 2R + \mathrm{O}(1)$ throughout its range. Replacing s by $s+2R$ we get

$$S(R) = \sum_s \frac{(-1)^s (s+2R+1)!}{(s+2R-a-b-e)!(s-\alpha_e)!(s-\alpha_a)!(s-\alpha_b)!(\beta_e-s)!(\beta_a-s)!(\beta_b-s)!} \tag{2.13}$$

where $\alpha_a = a+c+f, \alpha_b = b+d+f, \alpha_e = c+d+e$, and $\beta_a = b+c+e+f, \beta_b = a+d+e+f$, $\beta_e = a+b+c+d$, and the range of s in (2.13) is $\max\{\alpha_a, \alpha_b, \alpha_e\} \leqslant s \leqslant \min\{\beta_a, \beta_b, \beta_e\}$.

Applying (1.18) to the ratio $(s+2R+1)!/(s+2R-a-b-e)!$ in (2.13) we obtain $S(R) \approx (2R)^{a+b+e+1} S_1, \{R^{-1}\}$, where

$$S_1 = \sum_s \frac{(-1)^s}{(s-\alpha_a)!(s-\alpha_b)!(s-\alpha_e)!(\beta_a-s)!(\beta_b-s)!(\beta_e-s)!}. \tag{2.14}$$

Let $\{x, y, z\} = \{a, b, e\}$ with $\alpha_x = \max\{\alpha_a, \alpha_b, \alpha_e\}$. With $s = \alpha_x + l$ we get

$$S_1 = \sum_{l=0}^{n} \frac{(-1)^{\alpha_x+l}}{l!(\alpha_x-\alpha_y+l)!(\alpha_x-\alpha_z+l)!(\beta_x-\alpha_x-l)!(\beta_y-\alpha_x-l)!(\beta_z-\alpha_x-l)!} \tag{2.15}$$

where $n+1$ is the number of terms in S_1.

Next let $S(a, b, e; d, c, f)$ denote the sum in (1.4). From (1.4) we have

$$\begin{aligned} &S(z, y, x; \alpha_x-\alpha_y+y-x, \alpha_z-\alpha_x+x-z, \alpha_y-\alpha_z+z-y) \\ &\quad = \sum_s (-1)^s [s!(\alpha_x-\alpha_y+s)!(\alpha_x-\alpha_z+s)!(z+y-x-s)! \\ &\qquad \times (z-y+x-\alpha_x+\alpha_y-s)!(y-z+x-\alpha_x+\alpha_z-s)!]^{-1} = (-1)^{\alpha_x} S_1. \end{aligned} \tag{2.16}$$

Indeed if x', y', and z' denote the edges of the coupling tetrahedron opposite to x, y, and z, respectively, then we have $\alpha_x = x+y'+z', \alpha_y = y+z'+x', \alpha_z = z+x'+y'$, and $\beta_x = y+y'+z+z'$, $\beta_y = z+z'+x+x'$, $\beta_z = x+x'+y+y'$. Hence the second equality in (2.16) follows from $z+y-x = \beta_x-\alpha_x$, $z-y+x+\alpha_y-\alpha_x = \beta_y-\alpha_x$, and $y+\alpha_z-\alpha_x+x-y = \beta_z-\alpha_x$.

Furthermore, using that $\alpha_x-\alpha_y+y-x = y'-x'$, $\alpha_y-\alpha_z+z-y = z'-y'$, $\alpha_z-\alpha_x+x-z = x'-z'$, and $z-y-(z'-y') = \beta_x-2(y-z')$, from (1.4) and (2.16) we obtain

$$\begin{aligned} (-1)^{\alpha_x+\beta_x} \begin{pmatrix} z & y & x \\ y'-x' & x'-z' & z'-y' \end{pmatrix} &= (z+y'-x')!(z-y'+x')!(y+x'-z')! \\ &\quad \times (y-x'+z')!(x+z'-y')!(x-z'+y')! S_1 \\ &= (a+c-f)!(a-c+f)!(b+f-d)!(b-f+d)! \\ &\quad \times (e+d-c)!(e-d+c)! S_1. \end{aligned} \tag{2.17}$$

Then from (2.12), (2.4), the fact that $S(R) = (1+\mathrm{O}(R^{-1}))S_1$, and (2.17) we obtain

$$\begin{aligned} \begin{Bmatrix} a & b & e \\ d+R & c+R & f+R \end{Bmatrix} &\approx (-1)^{a+b+e}(2R)^{-1/2} \\ &\quad \times \begin{pmatrix} x & y & z \\ y'-x' & x'-z' & z'-x' \end{pmatrix} \qquad \{R^{-1}\}. \end{aligned} \tag{2.18}$$

Formula (1.9) is the case $x = e$ of (2.18). □

3. Proof of theorem 2

We use a different approach to derive asymptotic formulae in theorem 2. It is based on an integral representation of the ${}_4F_3$ hypergeometric functions given in proposition 3.2.

We first prove a lemma for certain ${}_2F_1$ functions.

Lemma 3.1. *Let $a_1 \neq 0$, $a_2 \neq 0$, b_1, b_2, and c be real numbers, $c \neq -l$ for any $l \in \mathbb{N} \cup \{0\}$, and let $t \in \mathbb{C}$ be a complex number. Then*

$$E(x_1, x_2) := {}_2F_1\left(\begin{matrix} a_1x_1 + b_1, a_2x_2 + b_2 \\ c \end{matrix} \middle| \frac{t}{(a_1a_2x_1x_2)} \right) - {}_0F_1\left(\begin{matrix} - \\ c \end{matrix} \middle| t \right) = \mathrm{O}\left(\frac{1}{m}\right) \qquad (3.1)$$

as $m = \min\{x_1, x_2\} \to \infty$, in the sense that $E(x_1, x_2) \leqslant C_0/m$ for $m \geqslant m_0$, where C_0 and $m_0 \geqslant C_0$ depend on $|a_{1,2}|$, $|b_{1,2}|$, $\mathrm{dist}(c, \mathbb{Z})$, and $|t|$ only.

Proof. We write $E(x_1, x_2)$ in the form

$$E(x_1, x_2) = \sum_{k=1}^{\infty} \frac{1}{(c)_k k!} \left(\frac{(a_1x_1 + b_1)_k}{(a_1x_1)^k} \frac{(a_2x_2 + b_2)_k}{(a_2x_2)^k} - 1 \right) t^k =: \sum_{k=1}^{\infty} f_k(x_1, x_2) t^k. \qquad (3.2)$$

From [7], section 3.8 we have the asymptotic formula

$$\Gamma(y) = \left(\frac{y}{e}\right)^y \left(\frac{2\pi}{y}\right)^{1/2} (1 + \omega(y)) \qquad y \to \infty \qquad (3.3)$$

where $\omega(y) = (\frac{1}{12})y^{-1} + \mathrm{O}(y^{-2})$.

For $a > 0$ and large $x > 0$,

$$(ax + b)_k = \prod_{\nu=0}^{k-1} (ax + b + \nu) = \frac{\Gamma(ax + b + k)}{\Gamma(ax + b)}. \qquad (3.4)$$

From (3.4) and (3.3) we get

$$\frac{(ax + b)_k}{(ax)^k} = \frac{\Gamma(ax + b + k)}{\Gamma(ax + b)(ax)^k} = \left(1 + \frac{k}{ax + b}\right)^{ax+b-1/2} \times \left(1 + \frac{b + k}{ax}\right)^k \frac{1}{e^k} \left(\frac{1 + \omega(ax + b + k)}{1 + \omega(ax + b)}\right). \qquad (3.5)$$

Let $K \in \mathbb{N}$, $K = \mathrm{O}(x^\eta)$ as $x \to \infty$ with $\eta \in (0, \frac{1}{2})$. Let $a \neq 0$ and b be fixed. For $k \in \{0, 1, \ldots, K\}$ we have

$$\frac{(ax + b)_k}{(ax)^k} = \prod_{j=0}^{k-1} \left(1 + \frac{b + j}{ax}\right) = 1 + \frac{kb + k(k - 1)/2}{ax} + \sum_{s=2}^{k} \sum_{J \subset \{0,\ldots,k-1\},\, |J|=s} \frac{1}{(ax)^s} \prod_{j \in J} (b + j). \qquad (3.6)$$

Then for $k \in \{2, \ldots, K\}$ we get

$$\left| \frac{(ax + b)_k}{(ax)^k} - 1 - \frac{2bk + k(k - 1)}{2ax} \right| \leqslant \sum_{s=2}^{k} \binom{k}{s} \frac{(|b| + k)^s}{|ax|^s} = \frac{(|b| + k)^2}{|ax|^2} \sum_{l=0}^{k-2} \binom{k}{l + 2} \left(\frac{|b| + k}{|ax|}\right)^l \leqslant \frac{(|b| + k)^4}{|ax|^2} \sum_{l=0}^{k-2} \binom{k - 2}{l} \left(\frac{|b| + k}{|ax|}\right)^l = \frac{(|b| + k)^4}{|ax|^2} \left(1 + \frac{|b| + k}{|ax|}\right)^{k-2} \leqslant \frac{(|b| + k)^4}{|ax|^2} \mathrm{e}^{(|b|+k)(k-2)/|ax|} = \mathrm{O}\left(\frac{k^4}{x^2}\right) \qquad (3.7)$$

where we used the inequalities $\binom{k}{l+2} = k(k-1)/((l+1)(l+2))\binom{k-2}{l} \leqslant k^2\binom{k-2}{l}$, and $1+t \leqslant \mathrm{e}^t$ for $t \geqslant 0$.

From (3.6) and (3.7) we get

$$\sum_{k=0}^{K} f_k(x_1,x_2)t^k = \sum_{k=0}^{K} \frac{1}{(c)_k k!}\left(\left(\frac{b_1}{a_1x_1}+\frac{b_2}{a_2x_2}\right)k + \left(\frac{1}{a_1x_1}+\frac{1}{a_2x_2}\right)\frac{k(k-1)}{2} + \mathrm{O}\left(\frac{k^4}{m^2}\right)\right)t^k$$
$$= \left\{\left(\frac{b_1}{a_1x_1}+\frac{b_2}{a_2x_2}\right)\frac{t}{c}\,{}_0F_1\left(\begin{matrix}-\\ c+1\end{matrix}\,\middle|\, t\right) + \left(\frac{1}{a_1x_1}+\frac{1}{a_2x_2}\right)\frac{t^2}{2c(c+1)}\,{}_0F_1\left(\begin{matrix}-\\ c+2\end{matrix}\,\middle|\, t\right)\right\} + t^2\mathrm{O}\left(\frac{1}{m^2}\right) \tag{3.8}$$

where we used (4.4), and for the additional terms introduced with the ${}_0F_1$ expressions we used the estimate (see (3.3))

$$\sum_{k=K+1}^{\infty}\frac{|t|^k}{|(C)_k|k!} \leqslant \frac{|t|^{K+1}}{d^2(K+1)!}\sum_{l=0}^{\infty}\frac{|t|^l}{(K+2)_l} \leqslant \frac{|t|^K\mathrm{e}^{|t|}}{d^2K!} = \mathrm{O}\left(\frac{|t|\mathrm{e}}{K}\right)^{K-1/2} = \mathrm{O}\left(\frac{|t|^2}{m^2}\right) \tag{3.9}$$

with $K=\mathrm{O}(m^\eta)$, $\eta\in(0,\frac{1}{2})$, and $d=\mathrm{dist}(C,\mathbb{Z})>0$.

The O symbol in (3.8) depends only on $a_{1,2}$, $b_{1,2}$, c, and $|t|$.

As in (3.9) for the sum $\sum_{k=K+1}^{\infty} f_k(x_1,x_2)t^k$ we get

$$\left|\sum_{k=K+1}^{\infty} f_k(x_1,x_2)t^k\right| \leqslant \sum_{k=K+1}^{\infty}\frac{|t|^k}{|(c)_k|k!} = \mathrm{O}\left(\frac{|t|^2}{m^2}\right) \tag{3.10}$$

for large m and $K=[m^{1/3}]$.

We set $l=[|c|]$, $l_1=l+4$, $X_1=|a_1|x_1/2$, $X_2=|a_2|x_2/2$, $X=\max\{X_1,X_2\}$, $Y=\min\{X_1,X_2\}$. Note that $Y=\mathrm{O}(m)$. For the sum of the first terms we have

$$\left|\sum_{k=K+1}^{\infty}\frac{(a_1x_1+b_1)_k}{(a_1x_1)^k}\frac{(a_2x_2+b_2)_k}{(a_2x_2)^k}\frac{t^k}{(c)_k k!}\right| \leqslant \frac{|t|}{d^2}\sum_{k=K}^{\infty}\frac{(|a_1|x_1+|b_1|)_k}{(|a_1|x_1)^k}\frac{(|a_2|x_2+|b_2|)_k}{(|a_2|x_2)^k}\frac{|t|^k k^{[|c|]+2}}{(k!)^2}$$
$$\leqslant \frac{|t|}{d^2}\sum_{k=K}^{\infty}\frac{4}{e^{2k}}\left(1+\frac{2k}{|a_1|x_1}\right)^{|a_1|x_1+|b_1|+k}\left(1+\frac{2k}{|a_2|x_2}\right)^{|a_2|x_2+|b_2|+k}\frac{|t|^k k^l}{\Gamma(k)^2}$$
$$\leqslant \frac{4|t|}{d^2}\sum_{k=K}^{\infty}\left(1+\frac{k}{X}\right)^{3X+k}\left(1+\frac{k}{Y}\right)^{3Y+k}\frac{|t|^k k^l}{k^{2k-1}}$$
$$\leqslant \frac{4|t|}{d^2}\int_K^{\infty}\left(1+\frac{s}{X}\right)^{6X}\left(1+\frac{s}{Y}\right)^{2s}(1+|t|)^s s^{l_1-2s}\,\mathrm{d}s =: \frac{2|t|}{d^2}\int_K^{\infty}\mathrm{e}^{F(s)}\,\mathrm{d}s \tag{3.11}$$

where we used (3.3) and (3.5). We also used that for a fixed $k>0$ the function $g(t)=t\ln(1+k/t)$ is increasing on $[0,\infty)$, since $g(0)=0$ and $g'(t)>0$ for $t>0$. This positivity of $g'(t)$ follows from $g'(t)=\ln(1+k/t)-k/(t+k)>0$ for small $t>0$, $g'(t)\to 0$ as $t\to\infty$, and $g''(t)=-k^2/(t(k+t)^2)<0$ for $t>0$. At the end we used the inequality $k^{2k-1}\geqslant s^{2s-4}$, for $s\in[k,k+1]$ and large k. We now have

$$F(s) = 6X\ln(1+s/X) + 2s\ln(1+s/Y) + s\ln(1+|t|) + (l_1-2s)\ln s.$$

For $u\geqslant 0$, $\ln(1+u)\leqslant u$, hence for $K\leqslant s\leqslant Y$, $K=[m^{1/3}]$ and large m,

$$F(s) \leqslant 6s + 2s\ln 2 + s\ln(1+|t|) + (l_1-2s)\ln s \leqslant -s\ln s.$$

For $s > Y$, $1 + s/Y \leqslant es/Y$, hence

$$F(s) \leqslant 6s + 2s(1 + \ln(s/Y)) + s\ln(1 + |t|) + (l_1 - 2s)\ln s$$
$$= s(8 - 2\ln Y + \ln(1 + |t|)) + l_1 \ln s \leqslant -s \ln Y$$

if m is large enough. Consequently,

$$\int_K^\infty \mathrm{e}^{F(s)}\,\mathrm{d}s \leqslant \int_K^Y s^{-s}\,\mathrm{d}s + \int_Y^\infty Y^{-s}\,\mathrm{d}s \leqslant YK^{-K} + Y^{-Y}/\ln Y = \mathrm{O}(YK^{-K}) = \mathrm{O}(m^{-2}). \tag{3.12}$$

Then (3.1) follows from (3.8), (3.10)–(3.12). □

Lemma 3.1 will be used to verify asymptotic formulae (1.11) and (1.13). We consider the 6-j symbol $\begin{Bmatrix} a+\lambda/2 & b & e+\lambda/2 \\ b+\lambda & a+\lambda/2 & f \end{Bmatrix}$. To find the corresponding ${}_4F_3$ (to be denoted by A_f) we use (1.14). In this case for $\lambda \geqslant 0$,

$$\alpha_1 = \max\{a+b+e+2\lambda, a+b+e+\lambda, 2b+f+\lambda, 2a+f+\lambda\} = a+b+e+2\lambda$$
$$\{\beta_1, \beta_2, \beta_3\} = \{2a+2b+2\lambda, a+b+e+f+2\lambda, a+b+e+f+\lambda\}$$

and by (1.14) we get

$$A_f = {}_4F_3\left(\left. \begin{matrix} -f, -f+\lambda, e-a-b, a+b+e+2\lambda+2 \\ \lambda+1, a+e-b-f+\lambda, b+e-a-f+\lambda \end{matrix} \right| 1 \right). \tag{3.13}$$

Proposition 3.2. *Let $f \in \mathbb{N}$, a, b, e, and λ be numbers for which A_f is well defined. Then*

$$A_f = \frac{f!(1-\lambda)_f}{(-a-e+b-\lambda+1)_f(-b-e+a-\lambda+1)_f}\,\frac{1}{2\pi\mathrm{i}}$$
$$\times \oint_{C_r} {}_2F_1\left(\left. \begin{matrix} -a-e+b-\lambda+1, -b-e+a-\lambda+1 \\ 1-\lambda \end{matrix} \right| t \right)$$
$$\times {}_2F_1\left(\left. \begin{matrix} e-a-b, a+b+e+2\lambda+2 \\ 1+\lambda \end{matrix} \right| t \right) t^{-f-1}\,\mathrm{d}t \tag{3.14}$$

where $C_r = \{t : |t| = r\}$ and $r < 1$.

Proof. We first assume that none of the denominator parameters is a negative integer. We set $p = (a+b+e)/2$.

For $k+1, f \in \mathbb{N}$, $k \leqslant f$, and $-A \notin \mathbb{N} \cup \{0\}$ we have

$$(A)_k = \frac{(A)_f}{(A+k)_{f-k}} = \frac{(-1)^k(-A-f+1)_f}{(-A-f+1)_{f-k}} \tag{3.15}$$

Then for every integer $f \geqslant 0$ we can write A_f in the form

$$A_f = \sum_{k=0}^{f} \frac{(-f)_k(\lambda-f)_k(-2p+2e)_k(2p+2\lambda+2)_k}{k!(1+\lambda)_k(2p-2b-f+\lambda)_k(2p-2a-f+\lambda)_k}$$
$$= \frac{f!(1-\lambda)_f}{(-2p+2b-\lambda+1)_f(-2p+2a-\lambda+1)_f}$$
$$\times \sum_{k=0}^{f} \frac{(-2p+2b-\lambda+1)_{f-k}(-2p+2a-\lambda+1)_{f-k}(-2p+2e)_k(2p+2\lambda+2)_k}{(f-k)!(1-\lambda)_{f-k}k!(1+\lambda)_k}. \tag{3.16}$$

Therefore,

$$\sum_{f=0}^{\infty} A_f \frac{(-2p+2b-\lambda+1)_f(-2p+2a-\lambda+1)_f}{f!(1-\lambda)_f} t^f$$
$$= {}_2F_1\left(\begin{matrix} -2p+2b-\lambda+1,\ -2p+2a-\lambda+1 \\ 1-\lambda \end{matrix} \middle| t \right)$$
$$\times {}_2F_1\left(\begin{matrix} -2p+2e,\ 2p+2\lambda+2 \\ 1+\lambda \end{matrix} \middle| t \right). \tag{3.17}$$

Integrating this generating function on C_r and using the Cauchy formula we establish the integral representation (3.14). We then remove the assumptions on the denominator by analytic continuation. □

Proposition 3.3. *Let $a, b, e, f \to \infty$, $f \in \mathbb{N}$ so that $m = \min\{p-a, p-b, p-e\} \to \infty$ and $f = o(m^{1/2})$, and λ fixed, be numbers for which A_f is well defined. Let $\rho = p(p-e)/((p-a)(p-b))$. Then*

$$\frac{A_f}{f!(1-\lambda)_f} = \frac{1}{f!(1-\lambda)_f} {}_2F_1\left(\begin{matrix} -f, \lambda-f \\ 1+\lambda \end{matrix} \middle| -\rho \right)\left(1+\mathrm{O}\left(\frac{f^2}{m}\right)\right). \tag{3.18}$$

Furthermore, the O *term in (3.18) does not depend on f or m.*

Proof. From the integral representation (3.14) with $r = 1/(4(p-a)(p-b))$, we obtain

$$A_f(-2p+2b-\lambda+1)_f(-2p+2a-\lambda+1)_f r^f$$
$$= \frac{f!(1-\lambda)_f}{2\pi} \int_0^{2\pi} {}_2F_1\left(\begin{matrix} -2p+2b-\lambda+1, -2p+2a-\lambda+1 \\ 1-\lambda \end{matrix} \middle| r\mathrm{e}^{\mathrm{i}\theta} \right)$$
$$\times {}_2F_1\left(\begin{matrix} -2p+2e, 2p+2\lambda+2 \\ 1+\lambda \end{matrix} \middle| r\mathrm{e}^{\mathrm{i}\theta} \right) \mathrm{e}^{-\mathrm{i}f\theta}\, \mathrm{d}\theta. \tag{3.19}$$

Observe that the coefficient of t^f in the power series expansion of

$${}_0F_1\left(\begin{matrix} - \\ 1-\lambda \end{matrix} \middle| t \right) {}_0F_1\left(\begin{matrix} - \\ 1+\lambda \end{matrix} \middle| -\rho t \right) \tag{3.20}$$

equals

$$\sum_{k=0}^{f} \frac{(-\rho)^k}{k!(1+\lambda)_k(f-k)!(1-\lambda)_{f-k}} = \frac{1}{f!(1-\lambda)_f} {}_2F_1\left(\begin{matrix} -f, \\ l-f \\ 1+\lambda \end{matrix} \middle| -\rho \right)$$
$$= \frac{(1+\rho)^f}{f!(1-\lambda)_f} {}_2F_1\left(\begin{matrix} -f,\ 1+f \\ 1+\lambda \end{matrix} \middle| \frac{\rho}{\rho+1} \right)$$
$$= \frac{(1+\rho)^f}{(1+\lambda)_f(1-\lambda)_f} P_f^{(-\lambda,\lambda)}\left(\frac{1-\rho}{1+\rho}\right) \tag{3.21}$$

where we used (3.15) with $A = -f$ and $A = \lambda - f$ for the first equality, the Pfaff–Kummer transformation (1.16) for the second equality, and the hypergeometric series representation for the Jacobi polynomials from [10],

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)^n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right). \tag{3.22}$$

Let $H_1(re^{i\theta})$ and $H_2(re^{i\theta})$ denote respectively the first and the second ${}_2F_1$ appearing in (3.19), and $h_1(e^{i\theta})$ and $h_2(-\rho e^{i\theta})$ denote respectively the first and the second ${}_0F_1$ appearing in (3.20) with $t = e^{i\theta}$. We have

$$\begin{aligned}\left|\int_0^{2\pi} H_1(re^{i\theta})H_2(re^{i\theta})e^{-if\theta}\,d\theta - \int_0^{2\pi} h_1(e^{i\theta})h_2(-\rho e^{i\theta})e^{-if\theta}\,d\theta\right| \\ = \left|\int_0^{2\pi}\{(H_1(re^{i\theta}) - h_1(e^{i\theta}))H_2(re^{i\theta}) + (H_2(re^{i\theta}) \right. \\ \left. -h_2(-\rho e^{i\theta}))h_1(e^{i\theta})\}e^{-if\theta}\,d\theta\right| \\ \leqslant \frac{M_1+M_2}{m} = O\left(\frac{1}{m}\right)\end{aligned} \tag{3.23}$$

uniformly in θ as $m \to \infty$, with

$$M_1 = \sup_{|z|=1,\, m\geqslant m_0} H_2(re^{i\theta}) \leqslant \sup_{|z|=1} h_2(-\rho z) + 1$$
$$M_2 = \sup_{|z|=1} h_1(z)$$

where in the expression for M_1, m_0 is the constant from lemma 3.1 for the ${}_2F_1$ series defining $H_2(re^{i\theta})$, and we have applied lemma 3.1.

Furthermore, from (3.7) we have

$$\frac{(4(p-a)(p-b))^f}{(-2p+2a-\lambda+1)_f(-2p+2b-\lambda+1)_f} = 1 + O\left(\frac{f^2}{m}\right). \tag{3.24}$$

Then (3.18) follows from (3.19), (3.20), (3.23), and (3.24). □

By (3.18), (3.21) and the identity for Jacobi polynomials (see [10]), $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ we also have

$$\frac{A_f}{f!(1-\lambda)_f} = \frac{(-1)^f(1+\rho)^f}{(1+\lambda)_f(1-\lambda)_f}P_f^{(\lambda,-\lambda)}\left(\frac{\rho-1}{\rho+1}\right)\left(1+O\left(\frac{f^2}{m}\right)\right). \tag{3.25}$$

From (3.25) and Darboux asymptotic formula for Jacobi polynomials [15]

$$\begin{aligned}P_n^{(\alpha,\beta)}(\cos\theta) &= (\pi n)^{-1/2}(\sin(\theta/2))^{-\alpha-1/2}(\cos(\theta/2))^{-\beta-1/2} \\ &\quad\times \cos((n+(\alpha+\beta+1)/2)\theta - (\alpha+1/2)\pi/2) + O(n^{-3/2})\end{aligned}$$

we obtain the following.

Corollary 3.4. *Under the assumptions of proposition 3.3, we have*

$$\begin{aligned}\frac{A_f}{f!(1-\lambda)_f} &= \frac{(-1)^f(1+\rho)^f}{(1+\lambda)_f(1-\lambda)_f}\frac{1}{\sqrt{f\pi}}\left(\sin\frac{\theta}{2}\right)^{-\lambda-1/2}\left(\cos\frac{\theta}{2}\right)^{\lambda-1/2} \\ &\quad\times\cos\left(\left(f+\frac{1}{2}\right)\theta - \left(\lambda+\frac{1}{2}\right)\frac{\pi}{2}\right)\left(1+O\left(\frac{f^2}{m}\right)+O\left(f^{-3/2}\right)\right)\end{aligned} \tag{3.26}$$

where $\theta = \arccos((\rho-1)/(\rho+1))$.

By the definition of $\rho = p(p-e)/((p-a)(p-b))$ and $\theta = \arccos((\rho-1)/(\rho+1))$ we have

$$\begin{aligned}\cos\theta &= \frac{p(p-e)-(p-a)(p-b)}{p(p-e)+(p-a)(p-b)} = \frac{2p(a+b-e)-2ab}{4p^2-2p(a+b+e)+2ab} \\ &= \frac{(a+b)^2-e^2-2ab}{2ab} = \frac{a^2+b^2-e^2}{2ab}.\end{aligned} \tag{3.27}$$

This expression agrees (asymptotically) with (1.12).

To verify (1.13) and (1.11) we set $\lambda = 0$ in (3.26) and the 6-j in the beginning of the section. From (1.5), (1.14) and (3.26) we get

$$\begin{aligned}
\begin{Bmatrix} a & b & e \\ b & a & f \end{Bmatrix} &= \Delta(a,b,e)^2\Delta(a,a,f)\Delta(b,b,f)S \\
&= \frac{(2p-2e)!(2p-2b)!(2p-2a)!}{(2p+1)!}\left(\frac{(2a-f)!f!(2b-f)!f!}{(2a+f+1)!(2b+f+1)!}\right)^{1/2} \\
&\quad\times\frac{(-1)^{a+b+c}(2p+1)!}{(2p-2b-f)!(2p-2a-f)!(2p-2e)!f!^2}A_f \\
&= (-1)^{a+b+e+f}\frac{(2p-2b-f+1)_f(2p-2a-f+1)!}{((2a-f+1)_{2f+1}(2b-f+1)_{2f+1})^{1/2}}\frac{(4ab)^f}{(2p-2a)^f(2p-2b)^f} \\
&\quad\times\frac{\cos((f+1/2)\theta-\pi/4)}{\left((\pi f/2)^{1/2}\right)}\left(1+\mathrm{O}\left(\frac{f^2}{m}\right)+\mathrm{O}\left(f^{-3/2}\right)\right) \\
&= \frac{(-1)^{a+b+e+f}\cos((f+1/2)\theta-\pi/4)}{\sqrt{\pi(2a-f+1)(2b-f+1)(f/2)\sin\theta}}\left(1+\mathrm{O}\left(\frac{f^2}{m}\right)+\mathrm{O}\left(f^{-3/2}\right)\right) \qquad (3.28)
\end{aligned}$$

if $f = \mathrm{o}(m^{1/2})$, where we applied (3.7) with $x = 2p-2a$, $2p-2b$, $2a$, and $2b$, and $k = f$ or $2f$ to the four ratios in the fourth line. In a main term sense (3.28) is the same as (1.13) or (1.11) with error of the ratio of order $\{f^2/m + f^{-3/2}\}$. In fact the first error term and the order of the second error term in these asymptotics formulae can be found explicitly using lemma 3.1. Our proof shows that (3.27) gives the right value of $\cos\theta$ in (1.11), and it is also simpler than (1.12).

4. Asymptotics for Racah polynomials

In deriving the asymptotics for the Racah polynomials (1.7) for a fixed $x \in \mathbb{N}$, we consider three cases when one or more of the integers (n), (N), $(N-n \geqslant 0)$ approaches infinity. For the last case we shall need another formula for the Racah polynomials which is obtained by applying the Whipple transformation (1.17) with $A = -x$, $B = n+\beta-N$, $C = x+\gamma+\delta+1$, $D = \gamma+1$, $E = \beta+\delta+1$, and $F = -N$:

$$\begin{aligned}
r_n(\lambda(x);\beta,\gamma,\delta) &= {}_4F_3\left(\begin{matrix} -n,\ n+\beta-N, -x, x+\gamma+\delta+1 \\ \beta+\delta+1, \gamma+1, -N \end{matrix}\ \middle|\ 1\right) \\
&= \frac{(\beta+\delta+x+1)_n(-N+x)_n}{(\beta+\delta+1)_n(-N)_n} \\
&\quad\times{}_4F_3\left(\begin{matrix} -n, -x, \gamma-\beta+N-n+1, -x-\delta \\ \gamma+1, -x-\beta-\delta-n, -x+N-n+1 \end{matrix}\ \middle|\ 1\right). \qquad (4.1)
\end{aligned}$$

Case 1. $N \to \infty$, n-fixed.

For $a, b,$ and $k \in \mathbb{N}$ fixed and $|N| \to \infty$ we have

$$\frac{(a+N)_k}{(b+N)_k} = \prod_{j=1}^{k}\left(1+\frac{a-b}{b+N+j-1}\right) = 1+\frac{(a-b)k}{N}+\mathrm{O}\left(\frac{1}{N^2}\right) \qquad (4.2)$$

since $1/(N+c) = 1/N - c/(N(N+c)) = 1/N + \mathrm{O}(N^{-2})$ as $|N| \to \infty$. Then from (1.7) we get

$$r_n(\lambda(x);\beta,\gamma,\delta) = {}_3F_2\left(\begin{matrix} -n, -x, x+\gamma+\delta+1 \\ \beta+\delta+1, \gamma+1 \end{matrix}\ \middle|\ 1\right)$$

$$-\frac{(n+\beta)}{N}\frac{nx(x+\gamma+\delta+1)}{(\beta+\delta+1)(\gamma+1)}\,{}_3F_2\left(\begin{matrix}-n+1,-x+1,-x+\gamma+\delta+2\\ \beta+\delta+2,\gamma+2\end{matrix}\,\middle|\,1\right)$$
$$+\mathrm{O}\left(\frac{1}{N^2}\right) \tag{4.3}$$

where we used the identity (with $s=1$)

$$\sum_{k=0}^{n}\frac{(-n)_k(a)_k(b)_k}{(e)_k(f)_k k!}(k-s+1)_s=\frac{(-n)_s(a)_s(b)_s}{(e)_s(f)_s}\,{}_3F_2\left(\begin{matrix}-n+s,a+s,b+s\\ e+s,\ f+s\end{matrix}\,\middle|\,1\right) \tag{4.4}$$

for $s=1,2,\ldots,n$, which follows from (1.1).

Case 2. $N\to\infty$, $n\to\infty$ and $N-n=m$ with m being a fixed positive integer.

For $k\leqslant m$ by (4.2) we have

$$\frac{(-n)_k}{(-N)_k}=\frac{(n-k+1)_k}{(n+m-k+1)_k}=1-\frac{mk}{n}+\mathrm{O}\left(\frac{1}{n^2}\right). \tag{4.5}$$

For $k\in\{m+1,\ldots,n\}$ we get

$$\frac{(-n)_k}{(-N)_k}=\frac{(n-k+1)_m}{(n+1)_m}=\prod_{j=1}^{m}\left(1-\frac{k}{n+j}\right)=1-k\sum_{j=1}^{m}\frac{1}{n+j}$$
$$+\sum_{s=2}^{m}(-k)^s\sum_{J\subset\{1,\ldots,m\},|J|=s}\prod_{j\in J}\frac{1}{n+j}$$
$$=1-\frac{mk}{n}+\mathrm{O}\left(\frac{k}{n^2}\right)+\sum_{s=2}^{m}(-k)^s\binom{m}{s}\left(\frac{1}{n^s}+\mathrm{O}\left(\frac{1}{n^{s+1}}\right)\right)$$
$$=1-\frac{mk}{n}+\mathrm{O}\left(\frac{k^2}{n^2}\right). \tag{4.6}$$

In the asymptotic formulae in this and the next case we shall use possibly nonterminating ${}_3F_2$ and ${}_2F_1$ series. We need the following estimate:

$$\sum_{k=n}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_{p-1})_k k!}=\mathrm{O}(n^{A-B}) \tag{4.7}$$

where none of the b_j's is a negative integer, and $A:=\sum_{j=1}^{p}a_j<B:=\sum_{j=1}^{p-1}b_j$. Indeed from (1.2) and (3.3) we have

$$(a)_k=\mathrm{O}\left(\frac{(a+k)^{a+k-1/2}}{\mathrm{e}^{a+k}}\right)=\frac{k^{a+k-1/2}}{\mathrm{e}^k}\mathrm{O}\left(\mathrm{e}^{-a}\left(1+\frac{a}{k}\right)^{a+k}\right)=\mathrm{O}\left(\frac{k^{a+k-1/2}}{\mathrm{e}^k}\right)$$

for $k\geqslant n\to\infty$. Then the sum (4.7) is bounded from above by an absolute constant times

$$\sum_{k=n}^{\infty}k^{A-B-1}\leqslant\int_{n-1}^{\infty}x^{A-B-1}\,\mathrm{d}x=\frac{(n-1)^{A-B}}{B-A}=\mathrm{O}(n^{A-B}).$$

From (1.7), (4.5), (4.6), (4.4), and (4.7) with $A-B=-m-1\leqslant-2$ we obtain

$$r_n(\lambda(x);\beta,\gamma,\delta)={}_3F_2\left(\begin{matrix}-m+\beta,-x,x+\gamma+\delta+1\\ \beta+\delta+1,\gamma+1\end{matrix}\,\middle|\,1\right)$$
$$-\frac{m}{n}\frac{(m+\beta)(-x)(x+\gamma+\delta+1)}{(\beta+\delta+1)(\gamma+1)}\,{}_3F_2$$
$$\times\left(\begin{matrix}-m+\beta+1,-x+1,x+\gamma+\delta+2\\ \beta+\delta+2,\gamma+2\end{matrix}\,\middle|\,1\right)+\mathrm{O}\left(\frac{1}{n^2}\right). \tag{4.8}$$

Case 3. $N \to \infty$, $n \to \infty$, and $N - n \to \infty$ simultaneously, and $n \sim N - n$, that is, $C^{-1} \leqslant n/(N-n) \leqslant C$ for some constant $C > 1$. We now assume that $\beta + \delta + 1 > 0$, $\gamma + 1 > 0$, and to assure convergence in the ${}_2F_1$'s appearing in the final asymptotics formula we also assume that $x > -(\delta + \gamma)/2$. Then $x + \beta + \delta > 0$. To find the asymptotics of $r_n(\lambda(x); \beta, \gamma, \delta)$ in this case we use (4.1) and several estimates that appear below.

We will use (1.18) to show that for a fixed $c > 0$ and large $n \in \mathbb{N}$,

$$\frac{(-n)_k}{(-n-c)_k} = 1 + \frac{ck}{n} + \mathrm{O}\left(\frac{k^2}{n^2}\right) \tag{4.9}$$

uniformly for $k = 0, 1, \ldots, n$. Indeed for $k \leqslant n/2$ we have

$$\begin{aligned}\frac{(-n)_k}{(-n-c)_k} &= \frac{(n-k+1)_k}{(n+c-k+1)_k} = \frac{\Gamma(n+1)\Gamma(n+c-k+1)}{\Gamma(n-k+1)\Gamma(n+c+1)}\\ &= n^{-c}(n-k)^c\left(1 - \frac{c(1+c)}{2n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right)\left(1 + \frac{c(c+1)}{2(n-k)} + \mathrm{O}\left(\frac{1}{(n-k)^2}\right)\right)\\ &= \left(1 - \frac{k}{n}\right)^c\left(1 + \frac{c(c+1)k}{2n(n-k)} + \mathrm{O}\left(\frac{1}{n^2}\right)\right) = 1 + \frac{ck}{n} + \mathrm{O}\left(\frac{k^2}{n^2}\right)\end{aligned}$$

where we used (1.18) with $M = n$ and $M = n - k$, and that $1/(n-k) \leqslant 2k/n$ for $k = 1, \ldots, [n/2]$.

For $k > n/2$, $\mathrm{O}(k/n) = \mathrm{O}(k^2/n^2) = \mathrm{O}(1)$, and since $(n-k+1)_k/(n+c-k+1)_k < 1$ we get (4.9) in this case as well.

Next, since $m \sim n$, then $m + k \sim n$, and from (1.18) we get

$$\begin{aligned}\frac{(m+a)_k}{(m+b)_k} &= \frac{\Gamma(m+k+a)\Gamma(m+b)}{\Gamma(m+a)\Gamma(m+k+b)} = \left(1 + \frac{k}{m}\right)^{a-b}\left(1 + \frac{(a-b)(a+b+1)k}{m(m+k)} + \mathrm{O}\left(\frac{1}{m^2}\right)\right)\\ &= 1 + \frac{(a-b)k}{m} + \mathrm{O}\left(\frac{k^2}{m^2}\right).\end{aligned} \tag{4.10}$$

From (4.9) with $c = x + \beta + \delta > 0$, and (4.10) with $a = \gamma - \beta + 1$ and $b = -x + 1$ we get

$$\frac{(-n)_k}{(-n-c)_k}\frac{(m+a)_k}{(m+b)_k} = 1 + \frac{ck}{n} + \frac{(a-b)k}{m} + \mathrm{O}\left(\frac{k^2}{n^2}\right). \tag{4.11}$$

Furthermore, using (1.18) we obtain

$$\begin{aligned}\frac{(\beta+\delta+x+1)_n}{(\beta+\delta+1)_n} &= \frac{\Gamma(\beta+\delta+x+1+n)\Gamma(\beta+\delta+1)}{\Gamma(\beta+\delta+1+n)\Gamma(\beta+\delta+x+1)}\\ &= \frac{n^x}{(\beta+\delta+1)_x}\left(1 + \frac{x(2\beta+2\delta+x+1)}{2n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\frac{(-N+x)_n}{(-N)_n} &= \frac{(m-x+1)_n}{(m+1)_n} = \frac{\Gamma(N-x+1)\Gamma(m+1)}{\Gamma(m-x+1)\Gamma(N+1)}\\ &= N^{-x}\left(1 + \frac{x(x-1)}{2N} + \mathrm{O}\left(\frac{1}{N^2}\right)\right)m^x\left(1 - \frac{x(x-1)}{2m} + \mathrm{O}\left(\frac{1}{m^2}\right)\right)\\ &= (m/N)^x\left(1 - \frac{x(x-1)n}{2mN} + \mathrm{O}\left(\frac{1}{n^2}\right)\right).\end{aligned} \tag{4.13}$$

Substituting (4.11)–(4.13) into (4.1) we obtain

$$r_n(\lambda(x); \beta, \gamma, \delta) = \frac{m^x n^x}{(m+n)^x}\frac{1}{(\beta+\delta+1)_x}$$

$$\times \left(1 - \frac{x(x-1)n}{2m(m+n)} + \frac{x(2\beta+2\delta+x+1)}{2n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right)$$
$$\times \left\{ {}_2F_1\left(\begin{matrix} -x, -x-\delta \\ \gamma+1 \end{matrix} \middle| 1 \right) + \left(\frac{(\beta+\delta+x)}{n} + \frac{(\gamma-\beta+x)}{m}\right) \frac{x(x+\delta)}{(\gamma+1)} \right.$$
$$\left. \times {}_2F_1\left(\begin{matrix} -x+1, -x-\delta+1 \\ \gamma+1 \end{matrix} \middle| 1 \right) + \mathrm{O}\left(\frac{1}{n^{2x+\delta+\gamma+1}}\right) \right\} \qquad (4.14)$$

where we used (4.7) with $A - B = -2x - \delta - \gamma - 1 < -1$.

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References

[1] Askey R A and Wilson J A 1982 A set of hypergeometric orthogonal polynomials *SIAM J. Math. Anal.* **13** 651–5
[2] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics: Theory and Application* (Reading, MA: Addison-Wesley)
[3] Biedenharn L C and Louck J D 1981 *The Racah–Wigner Algebra in Quantum Mechanics* (Reading, MA: Addison-Wesley)
[4] van Dam H 1965 *Quantum Theory of Angular Momentum* (New York: Academic)
[5] Braun P A 1978 WKB method for three-term recurrence relations and quasi-energies of an anharmonic oscillator *Theor. Math. Phys.* **37** 1070–81
[6] Geronimo J S and Smith D 1992 WKB (Liouville–Green) analysis of second order difference equations and applications *J. Approx. Theory* **69** 269–300
[7] Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic)
[8] Ponzano G and Regge T 1968 Semiclassical limits of Racah coefficients *Spectroscopic and Group Theoretical Methods in Physics* ed F Block *et al* (Amsterdam: North-Holland) pp 1–58
[9] Racah G 1942 Theory of complex spectra *Phys. Rev.* **61** 186–97
Racah G 1942 *Phys. Rev.* **62** 438–62
Racah G 1943 *Phys. Rev.* **63** 367–82
Racah G 1949 *Phys. Rev.* **76** 1352–65
[10] Rainville E D 1960 *Special Functions* (New York: Macmillan)
[11] Schulten K and Gordon R 1975 Exact recursive evaluation of 3-j and 6-j coefficients for quantum-mechanical coupling of angular momenta *J. Math. Phys.* **16** 1961–70
[12] Schulten K and Gordon R 1975 Semiclassical approximations to 3-j and 6-j coefficients for quantum-mechanical coupling of angular momenta *J. Math. Phys.* **16** 1971–88
[13] Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)
[14] Spigler R and Vianello M 1994 Discrete and continuous Liouville–Green–Olver approximations: A unified treatment via Volterra-Stieltjes integral equations *SIAM J. Math. Anal.* **25** 720–32
[15] Szegő G 1975 *Orthogonal Polynomials* (Providence, RI: American Mathematical Society)
[16] Wigner E P 1959 *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (New York: Academic)
[17] Whipple F J W 1926 Well-poised series and other generalized hypergeometric series *Proc. London Math. Soc.* **25** 525–44
[18] Wilson J A 1991 Asymptotics for the ${}_4F_3$ polynomials *J. Approx. Theor.* **66** 58–71
[19] Wilson M W 1970 On the Hahn polynomials *SIAM J. Math. Anal.* **1** 131–9
[20] Wulfman C E 1971 Dynamical groups in atomic and molecular physics *Group Theory and Its Applications* vol 2 (New York: Academic) 145–97